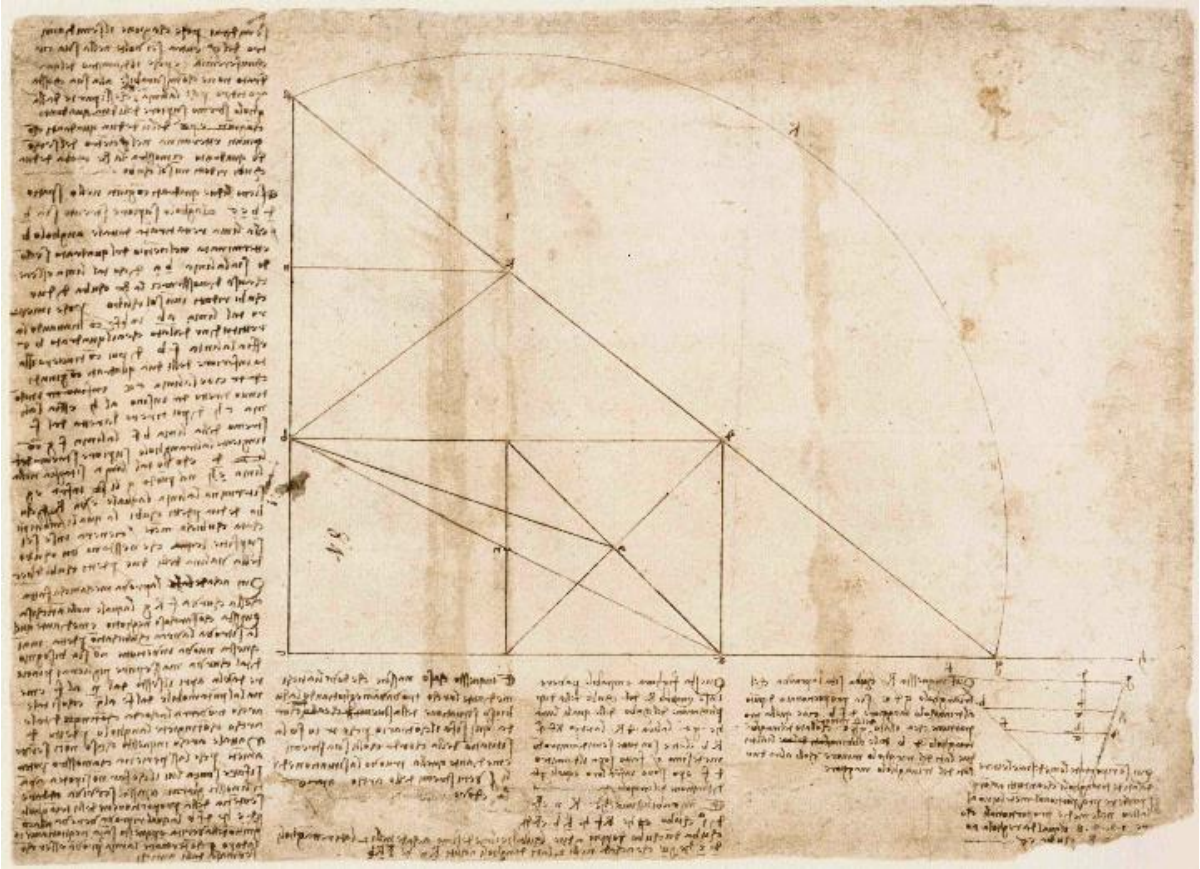


Leonardo and the Delian Problem

WITH TRANSLATION AND NOVEL SOLUTION

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Abstract

This paper includes a full English translation of Leonardo's famous straightedge and compass solution to the duplication of the cube found on Folio 588v (ca 1505) in the Codex Atlanticus. Via a step-by-step construction, we'll see that Leonardo brilliantly pushed the limits of straightedge accuracy, but his lack of mathematical prowess prevented him from seeing neither the error of his assertion that "without doubt that my proof is confirmed by antiquity", nor that the Fibonacci sequence was at the base of his construction. Next, we'll turn our attention to an earlier root-five rectangle construction on Folio 874r which, amazingly, was merely one diagonal step away from simultaneously duplicating the cube and squaring the circle to a greater accuracy than Folio 588v. Finally, based on the root-two rectangle, we'll present our own novel, straightedge-compass solution to the Delian problem, which is not only more precise than Leonardo's, but is perhaps the most precise straightedge-compass construction to date.

Keywords: Mathematics, Leonardo Da Vinci, Codex Atlanticus, Duplication of the Cube, Squaring the Circle, Fibonacci Sequence, Delian Problem, Plato's Academy, Sacred Geometry

Leonardo and the Delian Problem

The Delian problem, rooted in ancient Greek geometry, is closely associated with Apollo and the island of Delos. It challenges the duplication of a cube's volume using only a compass and straightedge, essentially requiring the computation of the cube root of 2, a task unattainable with these basic tools. This dilemma is one of the trio of classical Greek mathematical problems considered unsolvable by such means, along with angle trisection and circle squaring.

Originating from a mythological command to the Delians, the problem involves enlarging Apollo's cubic altar to alleviate a plague. The Greeks' ability to double a square's area did not extend to three-dimensional shapes, leading to this famous geometric puzzle.

The problem gained historical significance, particularly during Apollo's worship and festivals on Chios. Hippocrates of Chios, an influential Athenian mathematician, was one of the first to approach this problem, likely influenced by his teacher Oenopides, a pioneer in theoretical geometry. Oenopides' strict rule of using only a compass and straightedge ironically made solving the cube duplication more challenging.

The Delian problem rose to prominence during the 430-426 BC plague in Athens, a period survived by both Hippocrates and Plato. Ancient texts link this plague to a delegation sent by Plato to Delos, including notable figures like Archytas, Eudoxus, and Menaechmus, who attempted to double Apollo's altar using methods beyond the straightedge-compass limitation. The exact target of the plague is uncertain, but the legend suggests a divine admonition for neglecting mathematics, an ironic notion if aimed at Plato's Academy, known for its scientific pursuits.

In Plato's Academy, the Delian problem was likely seen as a philosophical and spiritual exercise, a view that aligns with Plato's emphasis on dialectic processes. The introduction of this problem to the Academy, possibly by Oenopides or Hippocrates, served as a philosophical rite of passage, merging geometry with deeper spiritual and philosophical inquiries. Hippocrates, influenced by the memory of Apollo's plague, advocated for the pursuit of truth through theoretical mathematics to align with Apollo, the deity of knowledge and healing. Plato, born during this plague, continued this tradition in his Academy, as evidenced by the inscription at its entrance, emphasizing the importance of geometry.

Leonardo da Vinci also dedicated significant efforts to solving this problem, indicative of the ancient link between architecture and mathematics.[1] The Delians' architectural

[1] Duvernoy, S. (2008). Leonardo and Theoretical Mathematics. In: Duvernoy, S. (eds) Nexus Network Journal. Nexus Network Journal, vol 10,1. Birkhäuser Basel. https://doi.org/10.1007/978-3-7643-8728-0_5

challenge involved constructing a new cubic altar with double the volume of the original. Their initial incorrect approach of merely doubling the cube's sides underscored the era's lack of understanding in the algebraic calculation of irrational numbers. Leonardo's interest in this problem likely stemmed from his newfound passion for Euclid, three-dimensional geometry and stereometry, leading him to explore various solutions, ranging from simple empirical methods to classical approaches, reflecting his diverse scientific research techniques.

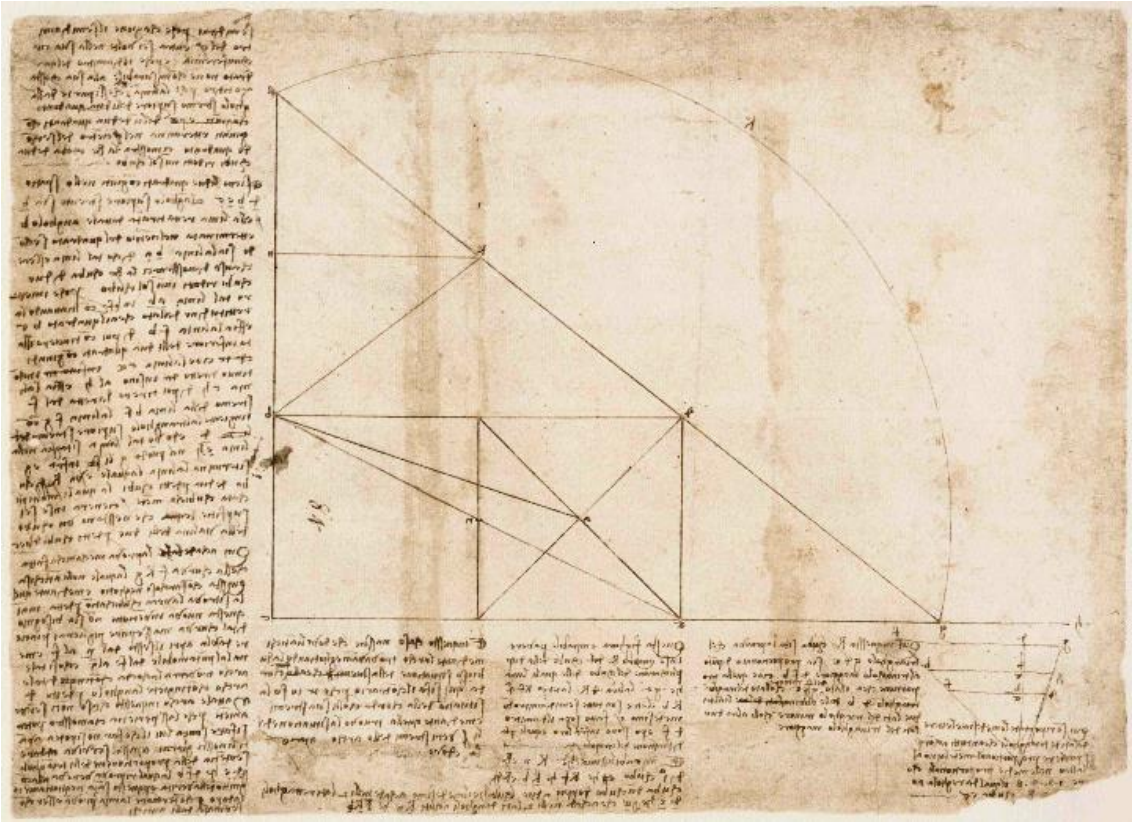


Figure 1 - Codex Atlanticus, Folio 588r (ca 1505, Pedretti, 1979)

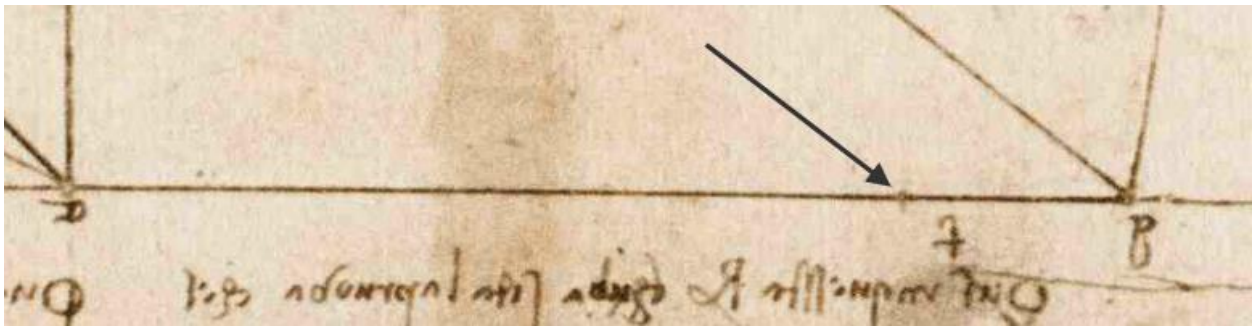


Figure 2 – Close up of Folio 588r

Folio 588 Translated

Figure with letters: fK-nK-bd-na-cegh

Using the cube root I will establish a proof that the smaller triangle GDE is proportionate with respect to the bigger triangle DFB. The proportion between side GE of the smaller triangle and side DB of the bigger triangle, in a way, follows the other two sides of both the smaller and bigger triangle. This marvelous figure contains four cubic roots concerning the duplication of the cube. The first is GE, the second is DK, the third is KF, and the fourth is KB. The last three are all radii of a circle with diameter DF, which means that they are equal to each other according to the definition of a circle. It is marvelous that … KN is a cubic root like KF and KB and the cubic root of a double cube with two cubes; the same applies to the two rectangular sides DE and GE, when compared to the two sides of KN and KB. This is where mathematics in antiquity faced the dubious situation that by means of an arc they had to wrestle with the questionable chord. Here, the situation is reversed - I create the situation with the chord at its extremes and use it to find the true ends of the arc, and so on. This is where… Lower right corner.

Triangle with the letters: fdc-bd-eg-fb-ed

Here the same letters are noted as the sides of the triangles that correspond proportionally to each other in the proportionate centers, such as 1, 2, 4, 8, whence any cube root can be derived, and so on.

Left column.

If you ask me the reason that the radius of a circle fits six times into its circumference, and that the diameter of a square cannot be measured against its side, then I will reply: "for the same reason that the straight line that starts out from the extreme upper corner of one of the two connected squares and ends at the center of the second square shows us the cubic root of a double cube reduced into just one cube". And there are two connected squares in the rectangle DBEC. From its point B in the extreme upper corner a straight line is drawn that ends in the center of the second square. This is line BA. I say that this line shows us how the cubic root of the double cube is reduced into just one cube. This is done by extending line BA along the side BC until line FB is equivalent to line BA. Then I extend the bottom of the two connected squares, i.e. the line CE, all the way past G (h). Then I draw a straight line from F (which is the end of the line BF), through the upper corner of D, to intersect the bottom line at point G. This is line FG. This also creates line EG which is the true cubic root of the two aforementioned squares, and which, if cubically multiplied, creates in itself six surfaces that represent a cube with a value double that of the cube of the two aforementioned squares contained in DBEC.

As shown, the mechanical proof happens by using the curve over FKG, which in antiquity was the arduous task used to find the true aforementioned cube calculation. In contrast, my new method can do without this curve. Whosoever out of interest should want to do it all the same may trace an arc the length of NF from point F to G. Thus the arc will determine that the chord FG contains point D. However, that arc is pointless because experience in this case shows that you can do without it, which was not possible in days gone by. It is used only to verify the truth and judge the proportion of the triangles GDE and DFB. Therefore, I deem without doubt that my proof is confirmed by antiquity.

First Construction

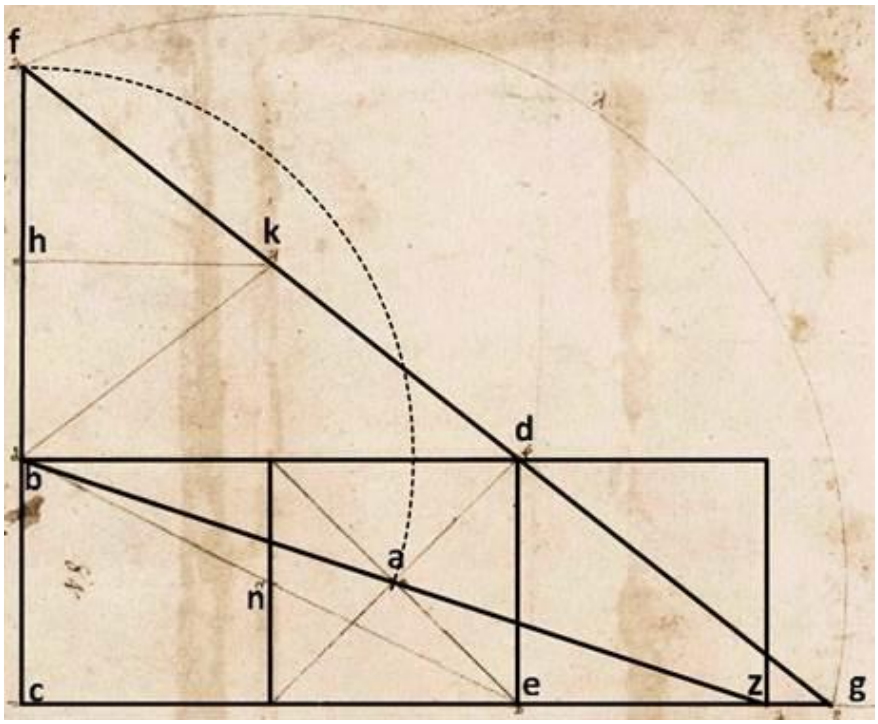


Figure 3

Not let's examine Leonardo's construction mathematically according to Figure 3. Note: Due to the need to put additional labeling, the lettered points do not correspond to Leonardo's labeling.

1. Create three adjacent squares, each with a side length of 1, so that line $\overline{CZ}$ is equal to 3.
2. Draw diagonal $\overline{BZ}$ and mark its midpoint as A.
3. Using the Pythagorean Theorem and the knowledge that $\overline{CB}$ = 1 and $\overline{CZ}$ = 3, determine the length of $\overline{BZ}$:

$$(1)^2 + (3)^2 = \left(\overline{BZ}\right)^2$$

$$\overline{BZ} = \sqrt{10} = 3.1622776601$$

4. Leonardo is clear that line $\overline{BA}$ is the length upon which he bases his construction for the doubling of the cube. Since $\overline{BA}$ is the midpoint of $\overline{BZ}$ we know that $\overline{BZ} = 1.5811388300$ for:

$$\frac{\sqrt{10}}{2} = 1.5811388300$$

5. With compass on point *B*, arc line $\overline{BZ}$ upward until it intersects the vertical axis, creating point *F*.

6. We can now determine the length of CF:

$$\overline{CF} = (\overline{CB} + \overline{BF}) = (1 + 1.5811388300) = 2.5811388300$$

7. From point *F* extend a line through point *D* to intersect the bottom horizontal axis. Label this intersection point *G*. Points *F*, *D* and *G* are collinear.

8. In order to check the validity of Leonardo's 'proof' we need to solve for triangle ΔFBD, which is similar to ΔDEG. First solve for angle $\angle BDF$:

$$tan\theta = \frac{\sqrt{10}}{2}\Big/_{2}$$

$$\angle BDF = \theta = 38.3288181°$$

9. Now, since $\angle BDF$ is congruent with $\angle CGF$, we can find the length of side $\overline{CG}$:

$$\tan(38.3288181) = \frac{\frac{\sqrt{10}}{2} + 1}{\overline{CG}}$$

$$\overline{CG} = 3.264911064$$

10. And because we know $\overline{CG}$ and $\overline{CE}$ we can now find the length of segment $\overline{EG}$

$$\overline{EG} = (\overline{CG} - \overline{CE}) = 1.264911064$$

11. We can now find the length of $\overline{DG}$ using the Pythagorean theorem:

$$(\overline{ED}^2 + \overline{EG}^2 = \overline{DG}^2) = (1^2 + 1.264911064^2 = 1.61245155^2)$$

$$\overline{DG} = 1.61245155$$

12. We are now in a position to validate (or invalidate) Leonardo's claim that his straightedge-compass construction is in accordance with Apollonius' dual-mean proportional solution. According to Apollonius we know that $\overline{BD}/\overline{BF} = \overline{BF}/\overline{EG} = \overline{EG}/\overline{DE}$, but we see that according to Leonardo's construction:

$$\overline{BD}/\overline{BF} = 1.264911064$$

$$\overline{BF}/\overline{EG} = 1.25$$

$$\overline{EG}/\overline{DE} = 1.264911064$$

Since the three terms in Leonardo's construction are not equivalent, it becomes evident that Leonardo's assertion of having streamlined Apollonius' technique while preserving its precision is unfounded. Although his construction undoubtedly offers a straightforward and rough approximation to the cube root of 2, he did not possess the mathematical expertise to validate it. However it interesting to note an apparent relationship to the Fibonacci sequence in Leonardo's construction, for:

$$\overline{DE} = \sqrt{\frac{5}{5}} = 1$$

$$\overline{EG} = \sqrt{\frac{8}{5}} = 1.26491106$$

$$\overline{DG} = \sqrt{\frac{13}{5}} = 1.61245155$$

$$\overline{BF} = \sqrt{\frac{5}{2}} = 1.58113883$$

$$\overline{BD} = \sqrt{\frac{8}{2}} = 2$$

$$\overline{FD} = \sqrt{\frac{13}{2}} = 2.54950975$$

Second Construction

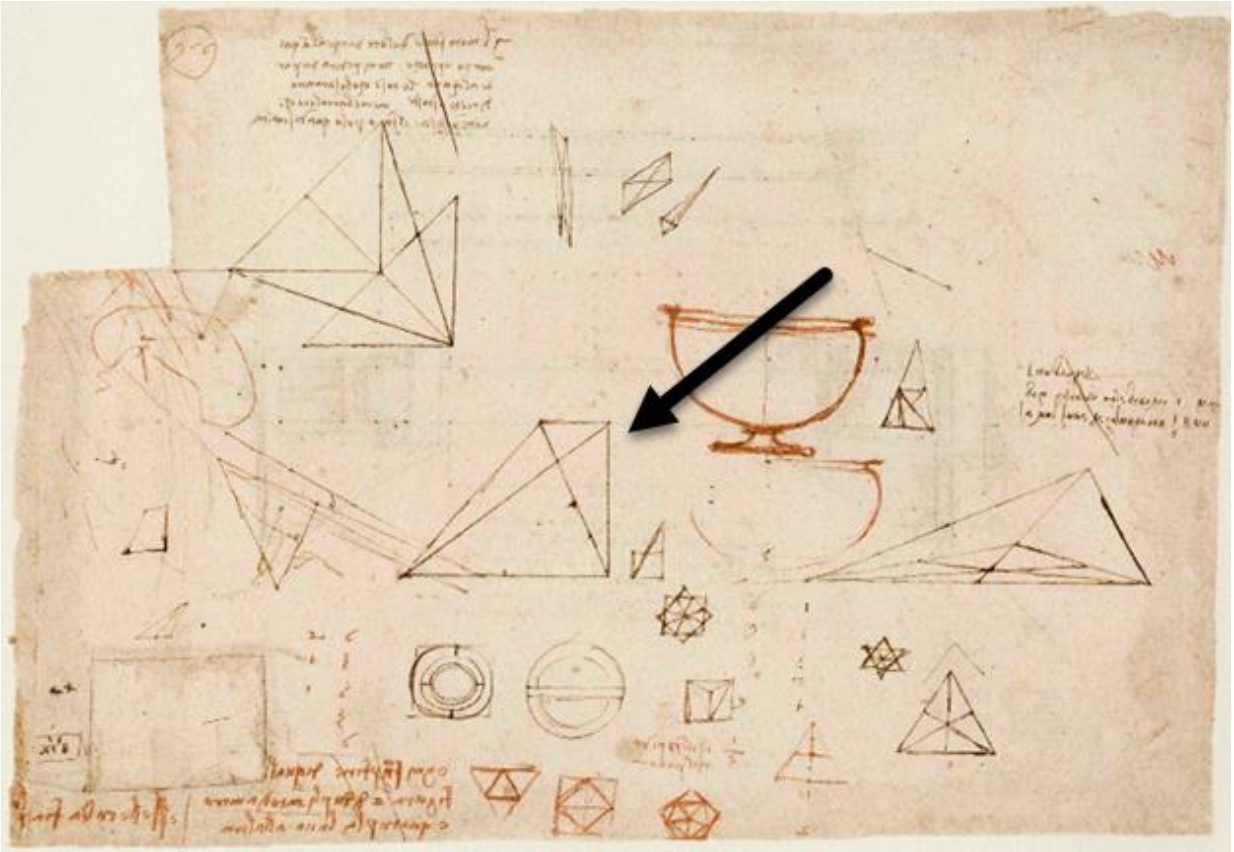


Figure 4 - Atlanticus, Folio 874v (ca January, 1496, Pedretti 1979)

Moving forward, we'll explore and complete an additional, uncomplicated design that Leonardo drafted but left without explanation (Figure 4). Found in the Atlanticus Folio 874v, this construction, when finished, seems to accurately approximate both Pi and the cube root of 2. It appears Leonardo may have been riffing on the bisection of a square in accordance with Euclid Book 1, Prop 10. Our analysis will start with the construction of a simple square ABJF (Figure 5).

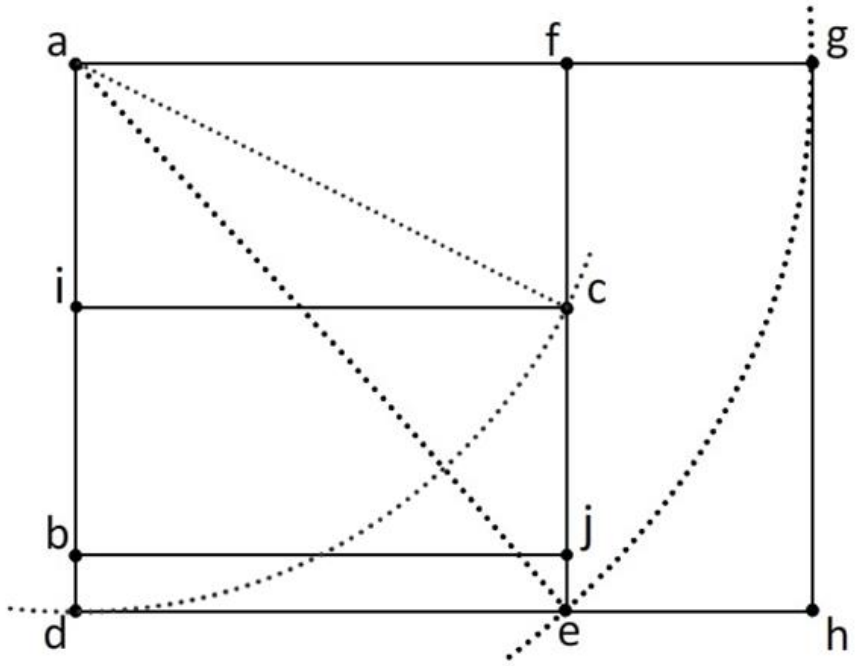


Figure 5

1. Create the square *ABJF*.

$$\overline{AB} = 1$$

2. Using Euclid Book 1, Prop 10 (not shown) bisect line $\overline{AB}$ and $\overline{FJ}$ to divide the square in half, creating the line $\overline{IC}$. Draw the diagonal $\overline{AC}$.

$$\overline{AC} = {}^{\sqrt{5}}/_{2} = 1.1180339887$$

3. With your compass on point *A*, arc line $\overline{AC}$ downward, through the extension of $\overline{AB}$. Label the intersection as point *D,* creating line $\overline{AD}$. Mirror the process from point *F* using the diagonal $\overline{FI}$, creating point *E* and line $\overline{FE}$.

$$\overline{AC} = \overline{FI} = \overline{AD} = \overline{FE} = 1.118033989$$

4. Draw line $\overline{DE}$, creating a new rectancgle *ADEF*.

5. Keep building. Again with your compass on point *A* arc the diagonal $\overline{AE}$ of rectangle *ADEF* upward through the extension of $\overline{AF}$. Label the intersection as point *G*, creating line $\overline{AG}$. Mirror the process from point *D* using the diagonal $\overline{DF}$, creating point *H* and line $\overline{DH}$.

$$(af^2 + ad^2 = ae^2) = (1^2 + 1.118033989^2 = 1.5^2)$$

$$\overline{AE} = \overline{DF} = \overline{AG} = \overline{DH} = 1.5$$

6. Draw line $\overline{GH}$, creating rectangle *ADHG*.

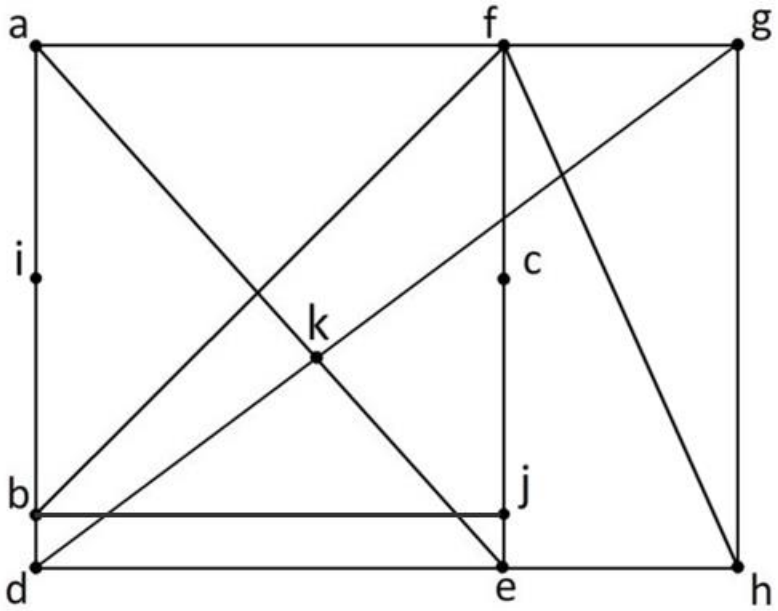


Figure 6

7. Consider the above Figure 6. Based upon our original $\overline{AC} = {}^{\sqrt{5}}/_{2}$ we have created four rectangles *ABJF, ADEF, ADHG and FGEH* and their diagonals respectively:

$$\overline{FB} = \sqrt{2} = 1.4142135623$$
$$\overline{AE} = {}^{\sqrt{9}}/_{2} = 1.5$$
$$\overline{GD} = {}^{\sqrt{14}}/_{2} = 1.8708286933$$
$$\overline{FH} = {}^{\sqrt{6}}/_{2} = 1.2247448713$$

Now to find an approximation to ***pi*** and the ***cube root of 2*** we need to solve triangle $\Delta\, AGK$ formed by the intersection of $\overline{GD}$ and $\overline{AE}$, and then compare its sides to the diagonal $\overline{FB}$ of our original square.

8. To solve $\Delta\, AGK$, we need to first solve for angles $\angle\, DGA$ and $\angle\, FAE$ which are congruent to $\angle\, KGA$ and $\angle\, KAG$.

$$tan\theta = \frac{1.118033989}{1.5}$$
$$\angle\, DGA = \theta = 36.69922521° = \angle\, KGA$$
$$tan\theta_2 = \frac{1.118033989}{1}$$
$$\angle\, FAE = \theta_2 = 48.18968511° = \angle\, KAG$$

9. Now subtract the sum of $\angle\, DGA$ and $\angle\, KAG$ from 180° (the sum of the interior angles of a triangle) to find $\angle\, AKG$.

$$180° - (\angle\, DGA + \angle\, KAG) = 95.11108968°$$
$$\angle\, AKG = 95.11108968°$$

10. We can now use the *law of sines* to find that when compared to $\overline{FB}$, the two sides of $\Delta\, AGK$, amazingly, give a very close, compass-straightedge approximation to both the **cube root of two** and **pi/2**.

a. The side $\overline{GK}$ and the *cube root of 2* (doubling the cube).

$$1.5/_{\sin(\angle AKG)} = \overline{GK}/_{sin\theta_2}$$

$$\overline{GK} = 1.122497216$$

$$\overline{FB} = \sqrt{2}$$

$$\sqrt[3]{2} = 1.2599210\ldots$$

$$\boldsymbol{\overline{FB}/\overline{GK} = 1.2598815\ldots (accuracy\ 0.0039\%)}$$

b. The side $\overline{AK}$ and *pi/2* (squaring the circle).

$$\frac{1.5}{sin95.11108968°} = \frac{\overline{AK}}{sin36.69922521°}$$

$$\overline{AK} = 0.9$$

$$\overline{FB} = \sqrt{2}$$

$$\frac{\pi}{2} = 1.570796\ldots$$

$$\boldsymbol{\overline{FB}/\overline{AK} = 1.571348\ldots (accuracy\ 0.055\%)}$$

As a matter of practicality, the standard approximation for pi that Leonardo (and most renaissance engineers) used was 22/7:[2]

$$\frac{22/_7}{2} = 1.5714285\ldots (accuracy\ 0.063\%).$$

[2] See Forster III 70r

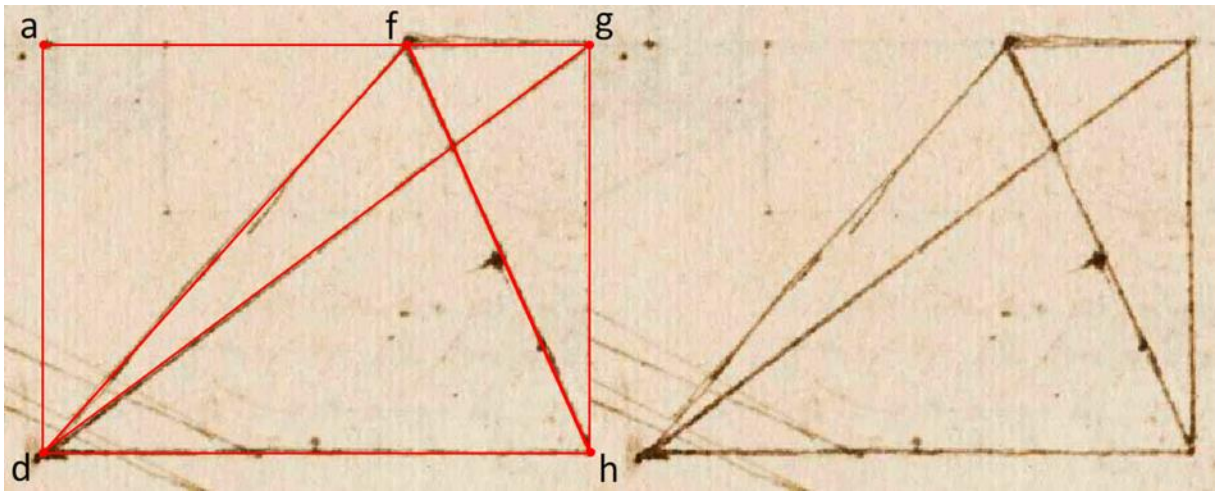


Figure 7

Now notice Figure 7. Compare the shape we constructed from Figure 6 (on left, in red) to the drawing done by Leonardo (on right) and notice the virtually exact fit.

Third Construction

A novel inverse Straightedge-Compass solution to doubling the cube, by halving a cube with side equal to √2, accuracy 0.0018% (Figure 8).

To demonstrate that $\overline{EC}/\overline{BN}$ is an very simple and accurate straightedge-compass solution for the $\sqrt[3]{2}$.

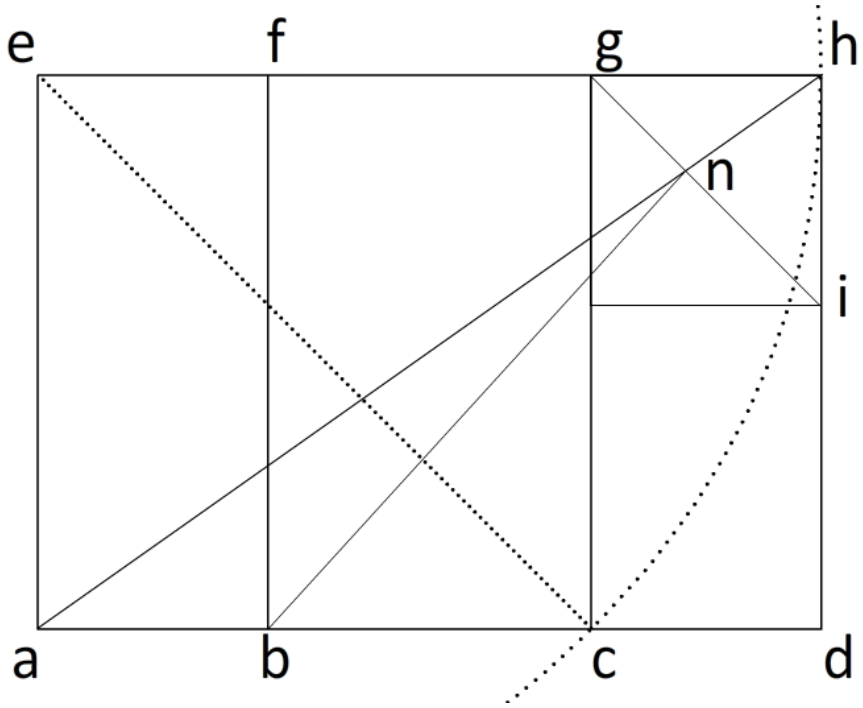


Figure 8

Steps of construction.

1) Create square *ACEG* with diagonal $\overline{EC}$. $\overline{EC}$ represents the side of a cube whose volume we are going to halve.

$$\overline{AE} = 1$$
$$\overline{EC} = \sqrt{2} = 1.4142135624$$

2) With your compass point on E swipe $\overline{EC}$ as an upward arc to create point H and line $\overline{EH}$. Then with compass on A swipe $\overline{AG}$ as a downward arc to create point D and line $\overline{AD}$.

$$\overline{EC} = \overline{EH} = \overline{AD} = \sqrt{2} = 1.4142135624$$

3) Create rectangle $ADEH$ and its diagonal by drawing lines $\overline{DH}$ and $\overline{AH}$.

$$\overline{AE}^2 + \overline{EH}^2 = \overline{AH}^2$$

$$\overline{AH} = \sqrt{3} = 1.7320508075$$

4) With compass on point H swipe line $\overline{HD}$ upward to create point F. Then with compass on point D swipe line $\overline{DH}$ downward to create point D. Draw line $\overline{FB}$ to create square $BDFH$.

$$\square BDFH = \square ACEG$$

$$\overline{HG} = \overline{AB} = (\overline{EH} - \overline{EG}) = (1.4142135624 - 1) = 0.4142135624$$

5) To create diagonal $\overline{GI}$ place your compass on point H and swipe line $\overline{HG}$ downward to create point I and line $\overline{HI}$. Draw line $\overline{GI}$.

$$\overline{HI} = \overline{HG} = 0.4142135624$$

$$(\overline{HI}^{\,2} + \overline{HG}^{\,2}) = \overline{GI}^{\,2}$$

$$\overline{GI} = 0.5857864376$$

6) Now we can draw line $\overline{BN}$. However, in order determine its length, we first need to solve for $\angle NAB$.

$$tan\theta = \frac{opposite}{adjacent} = \overline{HD}/\overline{AD} = \frac{1}{\sqrt{2}} \rightarrow \theta = 35.26438968°$$

7) Next, we need to solve for triangle ΔGHN, using the knowledge that:
   a. $\angle GHN$ is congruent to $\angle NAB$, and so is equal to 35.26438968°, and
   b. $\angle IGH$, because it is sqaure with $\angle IHG$, is 45°

$$\varphi = 180° - 45° - 35.26438968° = 99.73561032°$$

8) We're getting close. Using the *Law of Sines* we can now find the length of $\overline{AN}$ by finding $\overline{NH}$ and then subtracting $\overline{NH}$ from $\overline{AH}$.

$$\overline{GH}/_{sin\varphi} = \overline{NH}/_{sin45°} \rightarrow \overline{NH} = 0.2971729372$$

$$\overline{AN} = (\overline{AH} - \overline{NH}) \rightarrow \overline{AN} = 1.43487787$$

9) Now that we know the lengths of both $\overline{AB}$ and $\overline{AN}$ we can find the length of segment $\overline{BN}$ by using the *Law of Cosines*:

$$(\overline{BN})^2 = (\overline{AB})^2 + (\overline{AN})^2 - (2)(\overline{AB})(\overline{AN})cos\theta$$

$$\overline{BN} = 1.122445825$$

10) By constructing $\overline{EC}$ and $\overline{BN}$ with our straightedge-compass we have accurately and simply halved (or doubled) the cube. For:

$$\sqrt[3]{2} = 1.25992104 \ldots$$
$$\overline{EC} = \sqrt{2} = 1.41421356 \ldots$$
$$\overline{BN} = 1.122445825 \ldots$$
$$\overline{EC}/_{\overline{BN}} = \mathbf{1.2599392} \ldots (\boldsymbol{accuracy}\ \mathbf{0.0018}\%)$$